Resolvent expansions and continuity of the scattering matrix at embedded thresholds: the case of quantum waveguides

Serge Richard, Rafael Tiedra de Aldecoa

Resolvent expansions and continuity of the scattering matrix at embedded thresholds

S. Richard[1] and R. Tiedra de Aldecoa[2*]

[1] *Graduate school of mathematics, Nagoya University, Chikusa-ku, Nagoya 464-8602, Japan; On leave of absence from Université de Lyon; Université Lyon 1; CNRS, UMR5208, Institut Camille Jordan, 43 blvd du 11 novembre 1918, F-69622 Villeurbanne-Cedex, France.*

[2] *Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Av. Vicuña Mackenna 4860, Santiago, Chile*

E-mails: *richard@math.univ-lyon1.fr, rtiedra@mat.puc.cl*

Abstract

We present an inversion formula which can be used to obtain resolvent expansions near embedded thresholds. As an application, we prove for a class of quantum waveguides the absence of accumulation of eigenvalues and the continuity of the scattering matrix at all thresholds.

2010 Mathematics Subject Classification: 47A10, 81U35, 35J10.

Keywords: Thresholds, resolvent expansions, scattering matrix, quantum waveguides.

1 Introduction

During the recent years, there has been an increasing interest in resolvent expansions near thresholds and their various applications. These developments were partially initiated by the paper of A. Jensen and G. Nenciu [7] in which a general framework for asymptotic expansions is presented and then applied to potential scattering in dimension 1 and 2. The key point of that paper is an inversion formula which provides an efficient iterative method for inverting a family of operators $A(z)$ as $z \to 0$ even if $\ker\big(A(0)\big) \neq \{0\}$. Corrections or improvements of this inversion formula can be found in [2, Lemma 4], [5, Prop. 3.2] and [8, Prop. 1]. However, in all these papers either it is assumed that $A(0)$ is self-adjoint, or the construction relies on a Riesz projection which is not always convenient to deal with. These features are harmless in these works, since the threshold considered always lies at the endpoints of the spectrum of the underlying operator. However, once dealing with embedded thresholds, these features turn out to be critical.

Our aim in the present paper is thus twofold. On the one hand, we revisit the mentioned inversion formula, and on the other hand we show how its revised version can be used for proving the continuity of a scattering matrix at embedded thresholds. The abstract part of our results is presented in Section 2, and consists first in the derivation of the inversion formula without requiring that $A(0)$ is self-adjoint or that the projection is a Riesz projection (see Proposition 2.1). We then discuss two natural choices for the projection: either the Riesz projection defined in terms of the resolvent of $A(0)$ if 0 is an isolated point in the spectrum of $A(0)$, or the orthogonal projection on $\ker\big(A(0)\big)$ if $A(0)$ has a non-negative imaginary part. If both conditions hold, we also discuss the relations between these two projections, and provide

*Supported by the Chilean Fondecyt Grant 1130168 and by the Iniciativa Cientifica Milenio ICM RC120002 "Mathematical Physics" from the Chilean Ministry of Economy.

sufficient conditions for their equality. This situation often takes place in applications even without the assumption that $A(0)$ is self-adjoint (see Corollary 2.9).

In the second part of the paper (Section 3), we present an application of our abstract results to scattering theory for quantum waveguides. Quantum waveguides provide a particularly good model of study since their Hamiltonians possess an infinite number of embedded thresholds (with a change of multiplicity at each threshold) but give rise to a simple scattering theory taking place in a one-Hilbert space setting. We refer to [11] for basic results and earlier references on the spectral and scattering theory for quantum waveguides.

For a straight quantum waveguide with a compactly supported potential V, we derive an asymptotic expansion of the resolvent in a neighbourhood of each embedded threshold. More precisely, if the potential is written as $V = vuv$ with v non-negative and u unitary and self-adjoint, and if H_0 is the Dirichlet Laplacian for the waveguide, then we give an expansion of the operator $\big(u + v(H_0 - z)^{-1}v\big)^{-1}$ as z converges to any threshold z_0 (see Proposition 3.2). Note also that the operator $v(H_0 - z_0)^{-1}v$ (once properly defined) has a non-trivial imaginary part. This fact automatically prevents the use of any approach assuming the self-adjointness of $A(0)$, as mentioned above.

We then deduce two consequences of this asymptotic expansion. First, we prove in Corollary 3.3 that the possible point spectrum of the operator $H := H_0 + V$ does not accumulate at thresholds. Since the thresholds are the only possible accumulation points for such a model, we thus rule out this possibility. Second, we characterize for all scattering channels corresponding to the transverse modes of the waveguide the behavior of the scattering matrix for the pair $\{H, H_0\}$ at embedded thresholds. More precisely, we show that the scattering matrix is continuous at the thresholds if the channels we consider are already open, and that the scattering matrix has a limit from the right at the thresholds if a channel precisely opens at these thresholds (see Proposition 3.8 for a precise formulation of this result). Up to our knowledge, these types of results are completely new since the analysis of the behavior of a scattering matrix at embedded thresholds has apparently never been preformed. We also show the continuity of the scattering matrix at embedded eigenvalues which are not located at thresholds. But in this case, similar results were already known for other models, see for example [4, Prop. 10] or [12, Prop. 6.7.11] (see also [3] where propagation estimates at embedded thresholds are obtained for a Schrödinger operator with time periodic potential).

As a final comment, we stress that we fully describe all possible behaviors at thresholds since we do not assume any condition on the absence of bound states or resonances at thresholds. Based on the expressions obtained in this paper, a Levinson's type theorem for quantum waveguides could certainly be derived, and deserves further investigations.

Acknowledgements. The authors thank A. Jensen for useful discussions.

2 Asymptotic expansion

In this section, we first derive an extension of [8, Prop. 1] without specifying the nature of the projection, and then discuss two possible choices for this projection. The symbol $\mathcal{H}$ stands for an arbitrary Hilbert space with norm $\|\cdot\|$ and scalar product $\langle\cdot,\cdot\rangle$, and $\mathscr{B}(\mathcal{H})$ denotes the algebra of bounded operators on $\mathcal{H}$ with norm also denoted by $\|\cdot\|$.

Proposition 2.1. *Let $O \subset \mathbb{C}$ be a subset with 0 as an accumulation point. For each $z \in O$, let $A(z) \in \mathscr{B}(\mathcal{H})$ satisfy*

$$A(z) = A_0 + zA_1(z),$$

with $A_0 \in \mathscr{B}(\mathcal{H})$ and $\|A_1(z)\|$ uniformly bounded as $z \to 0$. Let also $S \in \mathscr{B}(\mathcal{H})$ be a projection such that:

(i) $A_0 + S$ is invertible with bounded inverse,

(ii) $S(A_0 + S)^{-1}S = S$.

Then, for $|z|$ small enough the operator $B(z) : S\mathcal{H} \to S\mathcal{H}$ defined by

$$B(z) := \frac{1}{z}\Big(S - S\big(A(z) + S\big)^{-1}S\Big) \equiv S(A_0 + S)^{-1}\Bigg(\sum_{j\geq 0}(-z)^j\big(A_1(z)(A_0 + S)^{-1}\big)^{j+1}\Bigg)S \tag{2.1}$$

is uniformly bounded as $z \to 0$. Also, $A(z)$ is invertible in $\mathcal{H}$ with bounded inverse if and only if $B(z)$ is invertible in $S\mathcal{H}$ with bounded inverse, and in this case one has

$$A(z)^{-1} = \big(A(z) + S\big)^{-1} + \frac{1}{z}\big(A(z) + S\big)^{-1}SB(z)^{-1}S\big(A(z) + S\big)^{-1}.$$

Proof. For $z \in O$ with $|z| > 0$ small enough, one has the following equalities:

$$\begin{aligned}
B(z) &= \frac{1}{z}\Big(S - S\big(A_0 + S + zA_1(z)\big)^{-1}S\Big)\\
&= \frac{1}{z}S\Big(1 - (A_0 + S)^{-1}\big(1 + zA_1(z)(A_0 + S)^{-1}\big)^{-1}\Big)S\\
&= \frac{1}{z}S\Bigg(1 - (A_0 + S)^{-1} - (A_0 + S)^{-1}\sum_{k\geq 1}\big(-zA_1(z)(A_0 + S)^{-1}\big)^k\Bigg)S\\
&= \frac{1}{z}\big(S - S(A_0 + S)^{-1}S\big) + S(A_0 + S)^{-1}\Bigg(\sum_{j\geq 0}(-z)^j\big(A_1(z)(A_0 + S)^{-1}\big)^{j+1}\Bigg)S.
\end{aligned}$$

So, the condition (ii) implies the second equality in (2.1). The second part of the claim is a direct application of the inversion formula [7, Lemma 2.1]. □

The choice of the projection S plays an important role in the previous proposition. For example, if 0 is an isolated point in the spectrum $\sigma(A_0)$ of A_0, a natural candidate for S is the Riesz projection associated with this value, which is the choice made in [2, 7, 8]. Another natural candidate is the orthogonal projection on the kernel of A_0. However, for both choices additional conditions are necessary in order to verify conditions (i) and (ii). Below, we first discuss the case of the Riesz projection and then the case of the orthogonal projection.

2.1 Riesz projection

In this section, we assume that 0 is an isolated point in $\sigma(A_0)$ and write S_r for the corresponding Riesz projection. In that case, $A_0S_r = S_rA_0 = S_rA_0S_r$ and $A_0 + S_r$ is invertible with bounded inverse (see [9, Chap. III.6.4]). The condition (ii) above, namely $S_r(A_0 + S_r)^{-1}S_r = S_r$, is more complicated to check. However, if one assumes that $A_0S_r = 0$, or the stronger condition that A_0 is self-adjoint, then the equalities $S_r(A_0 + S_r)^{-1} = S_r = (A_0 + S_r)^{-1}S_r$ hold, and thus condition (ii) is satisfied (note that in that case a small simplification takes place on the r.h.s. of (2.1)). However, the condition $A_0S_r = 0$ does not always hold since A_0S_r is in general only quasi-nilpotent [9, Sec. III.6.5]. Fortunately, the condition $A_0S_r = 0$ holds if A_0 has a particular form, as shown in the following lemma (which is an extension of [8, Prop. 2]).

Lemma 2.2. *Assume that $A_0 = X + iY$, with X, Y bounded self-adjoint operators and $Y \geq 0$, and suppose that 0 is an isolated point in $\sigma(A_0)$. Let S_r be the corresponding Riesz projection, and assume that $S_rA_0S_r$ is a trace-class operator. Then, $A_0S_r = S_rA_0 = 0$.*

Note that the trace-class condition is satisfied if, for instance, $S_r\mathcal{H}$ is finite dimensional.

Proof. Since S_r is a projection which commutes with A_0, one has $A_0 S_r = S_r A_0 = S_r A_0 S_r$. Therefore, if J is the operator in $S_r\mathcal{H}$ given by $J := S_r A_0 S_r$, then

$$\operatorname{Im}\big\langle S_r\varphi, JS_r\varphi\big\rangle = \operatorname{Im}\big\langle S_r\varphi, S_r A_0 S_r S_r\varphi\big\rangle = \operatorname{Im}\big\langle S_r\varphi, A_0 S_r\varphi\big\rangle \geq 0 \quad \text{for all } \varphi \in \mathcal{H},$$

or equivalently $\operatorname{Im}(J) \geq 0$ in $S_r\mathcal{H}$. Since J is quasi-nilpotent [9, Eq. (III.6.28)] and trace-class, and since quasi-nilpotent trace-class operators have trace 0 [10, p. 32], it follows that

$$0 = \operatorname{Tr}(J) = \operatorname{Tr}\big(\operatorname{Re}(J)\big) + i\operatorname{Tr}\big(\operatorname{Im}(J)\big).$$

This equality together with the inequality $\operatorname{Im}(J) \geq 0$ imply that $\operatorname{Im}(J) = 0$. Thus, J is self-adjoint and quasi-nilpotent, which means that $J = 0$. $\square$

We now list a series of consequences of the previous result.

Corollary 2.3. *Suppose that the assumptions of Lemma 2.2 are satisfied, then the conditions (i) and (ii) of Proposition 2.1 are verified for $S = S_r$.*

Corollary 2.4. *Suppose that the assumptions of Lemma 2.2 are satisfied, then $S_r\mathcal{H} = \ker(A_0)$.*

Proof. The inclusion $S_r\mathcal{H} \subset \ker(A_0)$ follows from the equality $A_0 S_r = 0$. For the other inclusion, we set $S_r^\perp := 1 - S_r$ and suppose by absurd that there exists $\varphi \in \ker(A_0) \setminus \{0\}$ such that $\varphi \notin S_r\mathcal{H}$. Then, one would have $S_r^\perp\varphi \neq 0$ since $S_r^\perp\varphi = 0$ implies $\varphi \in S_r\mathcal{H}$, which is a contradiction. In addition, one would have

$$A_0 S_r^\perp \varphi = A_0(1 - S_r)\varphi = A_0\varphi + A_0 S_r\varphi = 0,$$

and thus $S_r^\perp\varphi \in \ker(A_0)$. However, this would contradict the fact that A_0 is invertible in $S_r^\perp\mathcal{H}$, as proved for example in [9, Thm. III.6.17]. $\square$

We finally present a simple result which holds under the assumptions of Lemma 2.2, but can be proved in a slightly more general context. The norms and scalar products of the different Hilbert spaces are written with the same symbols.

Lemma 2.5. *Let $\mathcal{G}$ be an auxiliary Hilbert space, take $Z_n \in \mathscr{B}(\mathcal{H}, \mathcal{G})$, and assume that the sum $\sum_n Z_n^* Z_n$ is weakly convergent. Let also $A_0 = X + i\sum_n Z_n^* Z_n$, with X a bounded self-adjoint operator in $\mathcal{H}$, and suppose that S is a projection satisfying $A_0 S = 0$ and $SA_0 = 0$. Then, $Z_n S = 0$ and $SZ_n^* = 0$ for each n.*

Proof. Let $\varphi \in \mathcal{H}$. Then, the first identity follows from the equalities

$$\big\|Z_n S\varphi\big\|^2 \leq \big\langle S\varphi, \big(\textstyle\sum_n Z_n^* Z_n\big)S\varphi\big\rangle = \operatorname{Im}\big\langle S\varphi, \big(X + i\sum_n Z_n^* Z_n\big)S\varphi\big\rangle = \operatorname{Im}\big\langle S\varphi, A_0 S\varphi\big\rangle = 0,$$

and the second identity follows from the equalities

$$\big\|Z_n S^*\varphi\big\|^2 \leq \big\langle S^*\varphi, \big(\textstyle\sum_n Z_n^* Z_n\big)S^*\varphi\big\rangle = -\operatorname{Im}\big\langle S^*\varphi, \big(X - i\sum_n Z_n^* Z_n\big)S^*\varphi\big\rangle = -\operatorname{Im}\big\langle S^*\varphi, A_0^* S^*\varphi\big\rangle = 0.$$

$\square$

2.2 Orthogonal projection on the kernel

In this section, we assume from the beginning that $A_0 = X + iY$, with X, Y bounded self-adjoint operators and $Y \geq 0$. In that case, one has $\ker(A_0) = \ker(X) \cap \ker(Y) = \ker(A_0^*)$. Also, if S_o denotes the orthogonal projection on $\ker(A_0)$, the relations $XS_o = 0 = S_o X$, $YS_o = 0 = S_o Y$ and $A_0 S_o = 0 = S_o A_0$ hold. Thus, if one shows that $A_0 + S_o$ is invertible with bounded inverse, then the conditions (i) and (ii) of Proposition 2.1 would follow. So, we concentrate in the sequel on this invertibility condition.

Since A_0 is reduced by the orthogonal decomposition $\mathcal{H} = S_o\mathcal{H} \oplus (1 - S_o)\mathcal{H}$ and since A_0 is trivial in the subspace $S_o\mathcal{H}$, the operator $A_0 + S_o$ is invertible with bounded inverse if the restriction of A_0 to $S_o^\perp\mathcal{H} := (1 - S_o)\mathcal{H}$ is invertible with bounded inverse. However, since $A_0|_{S_o^\perp\mathcal{H}}$ has an inverse on $\mathrm{Ran}\left(A_0|_{S_o^\perp\mathcal{H}}\right) = \mathrm{Ran}(A_0)$, and since $\mathrm{Ran}(A_0)$ is dense in $S_r^\perp\mathcal{H}$ (because $\overline{\mathrm{Ran}(A_0)} = \ker(A_0^*)^\perp = \ker(A_0)^\perp = S_r^\perp\mathcal{H}$), the only remaining question concerns the boundedness of the inverse A_0^{-1} on $\mathrm{Ran}(A_0)$. So, the following question looks natural, but unfortunately we have not been able to answer it yet:

Question 2.6. *Assume that $A_0 = X + iY$, with X, Y bounded self-adjoint operators and $Y \geq 0$, and suppose that 0 is an isolated point in $\sigma(A_0)$. Then, is it true that A_0 is invertible in $\ker(A_0)^\perp$ with bounded inverse?*

In the following two lemmas, we exhibit conditions under which this question can be answered affirmatively.

Lemma 2.7. *Assume that $A_0 = X+iY$, with X, Y bounded self-adjoint operators and $Y \geq 0$, and suppose that 0 is an isolated point in $\sigma(A_0)$. Let S_r denote the corresponding Riesz projection, and assume that $S_rA_0S_r$ is a trace-class operator. Then, A_0 is invertible in $\ker(A_0)^\perp$ with bounded inverse if and only if S_r is an orthogonal projection.*

Before giving the proof, we recall that if S_r is an orthogonal projection, then it automatically follows from Corollary 2.4 that $S_r = S_o$.

Proof. Sufficient condition: Assume that S_r is an orthogonal projection (and thus equal to S_o). Since A_0 is invertible in $S_r^\perp\mathcal{H}$ with bounded inverse by [9, Thm. III.6.17], one infers that A_0 is invertible in $S_o^\perp\mathcal{H} = \ker(A_0)^\perp$ with bounded inverse.

Necessary condition: Suppose by absurd that S_r is not an orthogonal projection, or more precisely that $S_r^\perp\mathcal{H} \neq S_o^\perp\mathcal{H}$ (since we already know that $S_r\mathcal{H} = \ker(A_0) = S_o\mathcal{H}$ by Corollary 2.4). Then, if there exists $\varphi \in S_r^\perp\mathcal{H} \setminus \{0\}$ with $\varphi \notin S_o^\perp\mathcal{H}$, one has $S_o\varphi \neq 0$ and $S_o^\perp\varphi \neq 0$, and for any $z \in \mathbb{C} \setminus \{0\}$ with $|z|$ small enough

$$(A_0 - z)^{-1}\varphi = (A_0 - z)^{-1}S_o\varphi + (A_0 - z)^{-1}S_o^\perp\varphi.$$

Now, we know from [9, Thm. III.6.17] that the l.h.s. has a limit in $\mathcal{H}$ as $z \to 0$. But since $S_o\varphi \in \ker(A_0)$, the first term on the r.h.s. does not have a limit as $z \to 0$. Therefore, the second term on the r.h.s. neither has a limit as $z \to 0$, and thus the operator A_0 is not invertible in $S_o^\perp\mathcal{H} = \ker(A_0)^\perp$.

On the other hand, if there exists $\varphi \in S_o^\perp\mathcal{H} \setminus \{0\}$ with $\varphi \notin S_r^\perp\mathcal{H}$, one has $S_r\varphi \neq 0$ and $S_r^\perp\varphi \neq 0$, and for any $z \in \mathbb{C} \setminus \{0\}$ with $|z|$ small enough

$$(A_0 - z)^{-1}\varphi = (A_0 - z)^{-1}S_r\varphi + (A_0 - z)^{-1}S_r^\perp\varphi.$$

In this case, the second term on the r.h.s. does have a limit in $\mathcal{H}$ as $z \to 0$, but the first term on the r.h.s. does not. Therefore, the l.h.s. does not have a limit in $\mathcal{H}$ as $z \to 0$, and thus the operator A_0 is not invertible in $S_o^\perp\mathcal{H} = \ker(A_0)^\perp$.

Summing up, if $S_r^\perp\mathcal{H} \neq S_o^\perp\mathcal{H}$, then A_0 is not invertible in $S_o^\perp\mathcal{H} = \ker(A_0)^\perp$, which concludes the proof of the claim. ☐

Lemma 2.8. *Assume that $A_0 = X + iY$, with X, Y bounded self-adjoint operators and $Y \geq 0$. Suppose also that $A_0 = U + K$ with U unitary and K compact, or that A_0 is a finite-rank operator. Then, A_0 is invertible in $\ker(A_0)^\perp$ with bounded inverse.*

Proof. Recall that $\mathrm{Ran}\left(A_0|_{\ker(A_0)^\perp}\right) \equiv \mathrm{Ran}(A_0)$ is dense in $S_r^\perp\mathcal{H}$. So, the boundedness of the inverse of A_0 in $\ker(A_0)^\perp$ follows from the closed graph theorem [9, Thm. III.5.20] if $\mathrm{Ran}(A_0)$ is closed. But, this is verified under both conditions. Under the first condition, one has $A_0 = U + K = (1 + KU^{-1})U$ with KU^{-1} is compact. So, $(1 + KU^{-1})$ is Fredholm, and the image of $U\mathcal{H} = \mathcal{H}$ by $(1 + KU^{-1})$ is closed [1, Thm. 4.3.4]. And under the second condition, $\mathrm{Ran}(A_0)$ is finite-dimensional and thus closed. ☐

Under the assumptions of Lemma 2.8, the value 0 is an isolated point in $\sigma(A_0)$. Thus, the Riesz projection S_r is well defined, and one obtains the following by combining the two previous lemmas:

Corollary 2.9. *Suppose that the assumptions of Lemma 2.8 are satisfied. Then, $S_r = S_o$, and the conditions (i) and (ii) of Proposition 2.1 are verified for $S = S_r = S_o$.*

Proof. We know from Lemma 2.8 that A_0 is invertible in $\ker(A_0)^\perp$ with bounded inverse. Thus, it follows from Lemma 2.7 that $S_r = S_o$ and that the conditions (i) and (ii) of Proposition 2.1 are verified for $S = S_r = S_o$ if $S_r A_0 S_r$ is a trace-class operator. But, the operator $S_r A_0 S_r$ is clearly trace-class if A_0 is a finite-rank operator. On the other hand, if $A_0 = U + K$ with U unitary and K compact, then the isolated eigenvalue 0 is of finite multiplicity, $S_r\mathcal{H}$ is finite-dimensional [9, Remark III.6.23], and $S_r A_0 S_r$ is also trace-class. □

3 Quantum waveguides

We introduce in this section the model of quantum waveguide we use and recall some of its basics properties. Much of the material is borrowed from [11] to which we refer for further information.

We consider a bounded open connected set $\Sigma \subset \mathbb{R}^{d-1}$ with $d \geq 2$, and let $-\Delta_{\mathrm{D}}^{\Sigma}$ be the Dirichlet Laplacian on Σ acting in $\mathsf{L}^2(\Sigma)$. This operator has a purely discrete spectrum $\tau := \{\lambda_n\}_{n\geq1}$ consisting in eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots$ repeated according to multiplicity. The corresponding set of eigenvectors is denoted by $\{f_n\}_{n\geq1}$ and the corresponding set of one-dimensional orthogonal projections is denoted by $\{\mathcal{P}_n\}_{n\geq1}$. Sometimes, we omit for simplicity to stress that $n \geq 1$.

Consider now the straight waveguide $\Omega := \Sigma \times \mathbb{R}$ with coordinates (ω, x), the Hilbert space $\mathcal{H} := \mathsf{L}^2(\Omega)$, and the Dirichlet Laplacian $H_0 := -\Delta_{\mathrm{D}}^{\Omega}$ on Ω acting in $\mathcal{H}$. This operator decomposes as $H_0 = -\Delta_{\mathrm{D}}^{\Sigma} \otimes 1 + 1 \otimes P^2$ in $\mathsf{L}^2(\Sigma) \otimes \mathsf{L}^2(\mathbb{R})$, with $P := -i\,\partial_x$ the usual self-adjoint operator of differentiation in $\mathsf{L}^2(\mathbb{R})$. So, the spectrum $\sigma(H_0)$ of H_0 is purely absolutely continuous with $\sigma(H_0) = [\lambda_1, \infty)$, and each value $\lambda \in \tau$ is a threshold in $\sigma(H_0)$ with a change of multiplicity.

In the sequel, we also consider a perturbation of H_0 by a scalar potential. But, first we recall a few results about the resolvents $R^0(z) := (P^2 - z)^{-1}$ in $\mathsf{L}^2(\mathbb{R})$ and $R_0(z) := (H_0 - z)^{-1}$ in $\mathcal{H}$, with $z \in \mathbb{C} \setminus \mathbb{R}$. In the x-variable, these operators have kernels

$$R^0(z)(x, x') = \frac{i}{2\sqrt{z}}\,\mathrm{e}^{i\sqrt{z}|x-x'|}, \quad x, x' \in \mathbb{R}, \tag{3.1}$$

and

$$R_0(z)(x, x') = \frac{i}{2}\sum_n \frac{\mathrm{e}^{i\sqrt{z-\lambda_n}|x-x'|}}{\sqrt{z-\lambda_n}}\,\mathcal{P}_n, \quad x, x' \in \mathbb{R}, \tag{3.2}$$

with the convention that $\mathrm{Im}(\sqrt{z}) > 0$ for $z \in \mathbb{C} \setminus \mathbb{R}$.

In the following lemma, we recall some weighted estimates for $R^0(z)$ which complement the asymptotic expansion given in [7, Lemma 5.1]. For λ and ζ real, the first result allows us to use the abbreviated notation $R^0(\lambda + \zeta)$ instead of the longer notation $R^0(\lambda + \zeta + i0)$. We use the notations $\mathbb{C}_+ := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ and $\langle x\rangle := (1 + x^2)^{1/2}$, and we let Q denote the self-adjoint multiplication operator by the variable in $\mathsf{L}^2(\mathbb{R})$.

Lemma 3.1. *Fix $\varepsilon > 0$, take $\lambda \in \mathbb{R} \setminus (-\varepsilon, \varepsilon)$ and let $\zeta \in \overline{\mathbb{C}_+}$ with $|\zeta| < \varepsilon/2$.*

(a) If $s > 1/2$, then the limit $\langle Q\rangle^{-s} R^0(\lambda + \zeta)\langle Q\rangle^{-s} := \lim_{\delta\searrow0}\langle Q\rangle^{-s} R^0(\lambda + \zeta + i\delta)\langle Q\rangle^{-s}$ exists in $\mathscr{B}\big(\mathsf{L}^2(\mathbb{R})\big)$, and this limit is a Hilbert-Schmidt operator with Hilbert-Schmidt norm

$$\big\|\langle Q\rangle^{-s} R^0(\lambda + \zeta)\langle Q\rangle^{-s}\big\|_{\mathrm{HS}} \leq \mathrm{Const.}\,|\lambda|^{-1/2}.$$

(b) If $s > 3/2$, then

$$\big\|\langle Q\rangle^{-s}\big\{R^0(\lambda + \zeta) - R^0(\lambda)\big\}\langle Q\rangle^{-s}\big\|_{\mathrm{HS}} \leq \mathrm{Const.}\,|\zeta|\,|\lambda|^{-1/2},$$

where the constant may depend on ε but not on λ and ζ.

Proof. The first claim follows from (3.1). For the second one, one has to compute the integral kernel of $\langle Q\rangle^{-s}\big(R^0(\lambda+\zeta)-R^0(\lambda)\big)\langle Q\rangle^{-s}$, taking into account the following equalities with $y=|x-x'|$ and $x,x'\in\mathbb{R}$:

$$\frac{e^{i\sqrt{\lambda+\zeta}\,y}}{\sqrt{\lambda+\zeta}}-\frac{e^{i\sqrt{\lambda}\,y}}{\sqrt{\lambda}}=\frac{-\zeta}{\sqrt{\lambda}\sqrt{\lambda+\zeta}\,(\sqrt{\lambda+\zeta}+\sqrt{\lambda})}\,e^{i\sqrt{\lambda+\zeta}\,y}+\frac{1}{\sqrt{\lambda}}\big(e^{i\sqrt{\lambda+\zeta}\,y}-e^{i\sqrt{\lambda}\,y}\big)$$

and

$$\frac{1}{\sqrt{\lambda}}\big(e^{i\sqrt{\lambda+\zeta}\,y}-e^{i\sqrt{\lambda}\,y}\big)=\frac{i\zeta y}{2\sqrt{\lambda}}\int_0^1\frac{e^{i\sqrt{\lambda+s\zeta}\,y}}{\sqrt{\lambda+s\zeta}}\,\mathrm{d}s.$$

□

Now, we consider a self-adjoint operator $H:=H_0+V$, where $V\in\mathrm{L}^\infty(\Omega;\mathbb{R})$ is measurable with bounded support. We impose the boundedness of the support for simplicity, but we note that our results would also hold for potentials V decaying sufficiently fast at infinity (see for example the seminal papers [6, 7] for precise conditions on the decay of V at infinity).

Following the standard idea of decomposing the perturbation into factors, we define $v:\Omega\to\mathbb{R}$ and $u:\Omega\to\{-1,1\}$ by

$$v(\omega,x):=|V(\omega,x)|^{1/2}\quad\text{and}\quad u(\omega,x):=\begin{cases}1 & \text{if } V(\omega,x)\geq 0\\ -1 & \text{if } V(\omega,x)<0,\end{cases}\qquad (\omega,x)\in\Omega.$$

Then, we obtain the following symmetrized resolvent formula

$$(H-z)^{-1}=R_0(z)-R_0(z)\,v\big(u+vR_0(z)\,v\big)^{-1}vR_0(z),$$

which is equivalent to

$$uv(H-z)^{-1}vu=u-\big(u+vR_0(z)\,v\big)^{-1}. \tag{3.3}$$

As a consequence, deriving expansions in z for the resolvent $(H-z)^{-1}$ amounts to deriving expansions in z for the operator $\big(u+vR_0(z)v\big)^{-1}$, which we do in the following section.

3.1 Asymptotic expansion at embedded thresholds or eigenvalues

We derive in this section a suitable asymptotic expansion in z for the operator $\big(u+vR_0(z)v\big)^{-1}$. As a by-product, we show in particular the absence of accumulation of eigenvalues of H.

We consider $z=\lambda+\zeta$ with $\lambda\in\mathbb{R}$ and $\zeta\in\mathbb{C}_+$, and we adapt a convention used in [7] by setting

$$\kappa:=-i\sqrt{\zeta},$$

which implies that $\zeta=-\kappa^2$, $\mathrm{Re}(\kappa)>0$ and $\mathrm{Im}(\kappa)<0$. Also, we define the sets

$$\vartheta(\varepsilon):=\big\{\kappa\in\mathbb{C}\mid|\kappa|<\varepsilon,\mathrm{Re}(\kappa)>0\text{ and }\mathrm{Im}(\kappa)<0\big\},\quad\varepsilon>0.$$

Then, the main result of this section reads as follows:

Proposition 3.2. *Suppose that $V\in\mathrm{L}^\infty(\Omega;\mathbb{R})$ has bounded support, and let $\lambda\in\tau\cup\sigma_{\mathrm{p}}(H)$. Then, for $\kappa\in\vartheta(\varepsilon)$ with $\varepsilon>0$ small enough the operator*

$$\mathrm{M}(\lambda,\kappa):=\big(u+vR_0(\lambda-\kappa^2)\,v\big)^{-1}$$

belongs to $\mathscr{B}(\mathcal{H})$ and admits an asymptotic expansion in κ. The precise form of this expansion is given in equations (3.12) *and* (3.16) *below.*

Proof. First of all, observe that for each $\lambda \in \mathbb{R}$, $\varepsilon > 0$ and $\kappa \in \vartheta(\varepsilon)$, one has $\mathrm{Im}(\lambda - \kappa^2) \neq 0$, and thus $\mathrm{M}(\lambda,\kappa) \in \mathscr{B}(\mathcal{H})$. Therefore, the rest of the proof consists in deriving an asymptotic expansion for $\mathrm{M}(\lambda,\kappa)$ as $\kappa \to 0$. More precisely, we assume that $\kappa \in \vartheta(\varepsilon)$ and impose smallness conditions on ε. Also, we allow the value of ε to change from one line to another. For simplicity, we distinguish the cases $\lambda \in \tau$ and $\lambda \in \sigma_{\rm p}(H) \setminus \tau$, treating first the case $\lambda \in \tau$. All the operators defined below depend on the choice of λ, but for simplicity we do not mention this dependence.

(i) Assume that $\lambda \in \tau$, set $N := \{n \geq 1 \mid \lambda_n = \lambda\}$, and write $\mathcal{P} := \sum_{n \in N} \mathcal{P}_n$ for the corresponding orthogonal projection (of dimension greater or equal to 1). Then, there exists $\varepsilon > 0$ such that $(\lambda - \lambda_n) \in \mathbb{R} \setminus (-\varepsilon, \varepsilon)$ for all $n \notin N$. So, Lemma 3.1(a) applies, and one has for $s > 1/2$ and $\kappa \in \vartheta(\sqrt{\varepsilon/2})$ that

$$\big\|\langle Q\rangle^{-s} R^0(\lambda - \kappa^2 - \lambda_n)\langle Q\rangle^{-s}\big\| \leq \mathrm{Const.}\,|\lambda - \lambda_n|^{-1/2} \quad \text{for } n \notin N.$$

Since $\langle Q\rangle^{-s} R^0(-\kappa^2)\langle Q\rangle^{-s}$ also belongs to $\mathscr{B}\big(\mathrm{L}^2(\mathbb{R})\big)$, and since $v(1 \otimes \langle Q\rangle^s) \in \mathscr{B}(\mathcal{H})$, one infers from (3.1)-(3.2) that

$$\mathrm{M}(\lambda,\kappa) = \bigg(v\big(\mathcal{P} \otimes R^0(-\kappa^2)\big)v + u + \sum_{n \notin N} v\big(\mathcal{P}_n \otimes R^0(\lambda - \kappa^2 - \lambda_n)\big)v\bigg)^{-1}.$$

Thus, if one writes [7, Eq. (3.13)]

$$v\big(\mathcal{P} \otimes R^0(-\kappa^2)\big)v = \frac{1}{2\kappa} N_0 + N_1(\kappa), \tag{3.4}$$

with N_0 and $N_1(\kappa) \in \mathcal{O}(1)$ integral operators which kernels satisfy

$$N_0(\omega,x,\omega',x') = \sum_{n \in N} f_n(\omega)\, v(\omega,x)\, v(\omega',x')\, \overline{f_n(\omega')}, \quad (\omega,x),(\omega',x') \in \Omega,$$
$$N_1(0)(\omega,x,\omega',x') = -\frac{1}{2}\sum_{n \in N} f_n(\omega)\, v(\omega,x)\,|x - x'|\, v(\omega',x')\, \overline{f_n(\omega')}, \quad (\omega,x),(\omega',x') \in \Omega,$$

one obtains that

$$\mathrm{M}(\lambda,\kappa) = 2\kappa\big(N_0 + 2\kappa\, M_1(\kappa)\big)^{-1}, \tag{3.5}$$

with

$$M_1(\kappa) := N_1(\kappa) + u + \sum_{n \notin N} v\big(\mathcal{P}_n \otimes R^0(\lambda - \kappa^2 - \lambda_n)\big)\, v.$$

Moreover, one infers from [7, Lemma 5.1(i)] and Lemma 3.1(a) that $\|M_1(\kappa)\|$ is uniformly bounded as $\kappa \to 0$.

Now, in order to analyse further the operator (3.5), we set

$$I_0(\kappa) := N_0 + 2\kappa\, M_1(\kappa).$$

Then, (3.5) reads $\mathrm{M}(\lambda,\kappa) = 2\kappa\, I_0(\kappa)^{-1}$, and our goal reduces to derive an asymptotic expansion for $I_0(\kappa)^{-1}$ as $\kappa \to 0$. Since $I_0(0) = N_0$ is a finite-rank operator, 0 is not a limit point of $\sigma(N_0)$. Also, N_0 is self-adjoint, therefore the orthogonal projection S_0 on $\ker(N_0)$ is equal to the Riesz projection of N_0 associated with the value 0. We can thus apply Proposition 2.1, and obtain for $\kappa \in \vartheta(\varepsilon)$ with $\varepsilon > 0$ small enough that the operator $I_1(\kappa) : S_0\mathcal{H} \to S_0\mathcal{H}$ defined by

$$I_1(\kappa) := \sum_{j \geq 0} (-2\kappa)^j S_0 \big\{M_1(\kappa)\big(I_0(0) + S_0\big)^{-1}\big\}^{j+1} S_0 \tag{3.6}$$

is uniformly bounded as $\kappa \to 0$. Furthermore, $I_1(\kappa)$ is invertible in $S_0\mathcal{H}$ with bounded inverse satisfying the equation

$$I_0(\kappa)^{-1} = \big(I_0(\kappa) + S_0\big)^{-1} + \frac{1}{2\kappa}\big(I_0(\kappa) + S_0\big)^{-1} S_0 I_1(\kappa)^{-1} S_0 \big(I_0(\kappa) + S_0\big)^{-1}.$$

It follows that for $\kappa \in \vartheta(\varepsilon)$ with $\varepsilon > 0$ small enough, one has

$$\mathrm{M}(\lambda,\kappa) = 2\kappa\big(I_0(\kappa)+S_0\big)^{-1} + \big(I_0(\kappa)+S_0\big)^{-1} S_0 I_1(\kappa)^{-1} S_0 \big(I_0(\kappa)+S_0\big)^{-1}, \tag{3.7}$$

with the first term vanishing as $\kappa \to 0$.

To describe the second term of $\mathrm{M}(\lambda,\kappa)$ as $\kappa \to 0$, we recall the equality $\big(I_0(0)+S_0\big)^{-1}S_0 = S_0$, which (together with (3.6)) implies that

$$I_1(\kappa) = S_0 M_1(0) S_0 + \kappa\, M_2(\kappa),$$

with

$$\begin{aligned} M_2(\kappa) &:= \frac{1}{\kappa} S_0\big(M_1(\kappa)-M_1(0)\big)S_0 + \frac{1}{\kappa}\sum_{j\geq 1}(-2\kappa)^j S_0\big\{M_1(\kappa)\big(I_0(0)+S_0\big)^{-1}\big\}^{j+1} S_0 \\ &= S_0 N_2(\kappa) S_0 + \frac{1}{\kappa} S_0 \sum_{n\notin N} v\big\{\mathcal{P}_n \otimes \big(R^0(\lambda-\kappa^2-\lambda_n) - R^0(\lambda-\lambda_n)\big)\big\} v S_0 \\ &\quad - 2\sum_{j\geq 0}(-2\kappa)^j S_0\big\{M_1(\kappa)\big(I_0(0)+S_0\big)^{-1}\big\}^{j+2} S_0 \end{aligned} \tag{3.8}$$

and

$$N_2(\kappa) := \frac{1}{\kappa}\big(N_1(\kappa)-N_1(0)\big).$$

Then, we observe that [7, Lemma 5.1(i)] together with (3.4) imply that $N_2(\kappa)$ admits a finite limit as $\kappa \to 0$. Also, we note that Lemma 3.1(b) implies that the second term in (3.8) vanishes as $\kappa \to 0$. Therefore, $\|M_2(\kappa)\|_{\mathscr{B}(S_0\mathcal{H})}$ is uniformly bounded as $\kappa \to 0$.

Now, we know that $M_1(0)$ is the sum of the unitary and self-adjoint operator u, the self-adjoint and compact operator $N_1(0)$, and a compact operator with non-negative imaginary part. So, since S_0 is an orthogonal projection with finite-dimensional kernel, the operator $I_1(0) = S_0 M_1(0) S_0$ acting in the Hilbert space $S_0\mathcal{H}$ can also be written as the sum of a unitary and self-adjoint operator, a self-adjoint and compact operator, and a compact operator with non-negative imaginary part. Thus, Corollary 2.9 applies with S_1 the finite-rank orthogonal projection on $\ker\big(I_1(0)\big)$, and the iterative procedure of Section 2 can be applied to $I_1(\kappa)$ as it was done for $I_0(\kappa)$.

Thus, for $\kappa \in \vartheta(\varepsilon)$ with $\varepsilon > 0$ small enough, the operator $I_2(\kappa): S_1\mathcal{H} \to S_1\mathcal{H}$ defined by

$$I_2(\kappa) := \sum_{j\geq 0}(-\kappa)^j S_1\big\{M_2(\kappa)\big(I_1(0)+S_1\big)^{-1}\big\}^{j+1} S_1$$

is uniformly bounded as $\kappa \to 0$. Furthermore, $I_2(\kappa)$ is invertible in $S_1\mathcal{H}$ with bounded inverse satisfying the equation

$$I_1(\kappa)^{-1} = \big(I_1(\kappa)+S_1\big)^{-1} + \frac{1}{\kappa}\big(I_1(\kappa)+S_1\big)^{-1} S_1 I_2(\kappa)^{-1} S_1 \big(I_1(\kappa)+S_1\big)^{-1}.$$

This expression for $I_1(\kappa)^{-1}$ can now be inserted in (3.7) in order to get

$$\begin{aligned} \mathrm{M}(\lambda,\kappa) &= 2\kappa\big(I_0(\kappa)+S_0\big)^{-1} + \big(I_0(\kappa)+S_0\big)^{-1} S_0 \big(I_1(\kappa)+S_1\big)^{-1} S_0 \big(I_0(\kappa)+S_0\big)^{-1} \\ &\quad + \frac{1}{\kappa}\big(I_0(\kappa)+S_0\big)^{-1} S_0\big(I_1(\kappa)+S_1\big)^{-1} S_1 I_2(\kappa)^{-1} S_1 \big(I_1(\kappa)+S_1\big)^{-1} S_0 \big(I_0(\kappa)+S_0\big)^{-1}, \end{aligned} \tag{3.9}$$

with the first two terms bounded as $\kappa \to 0$.

Let us concentrate on the last term and check once more that the assumptions of Proposition 2.1 are satisfied. For that purpose, we recall that $\big(I_1(0)+S_1\big)^{-1}S_1 = S_1$, and observe that

$$I_2(\kappa) = S_1 M_2(0) S_1 + \kappa\, M_3(\kappa),$$

with

$$M_2(0) = S_0 N_2(0) S_0 - 2 S_0 M_1(0)\big(I_0(0)+S_0\big)^{-1} M_1(0) S_0 \qquad \text{and} \qquad M_3(\kappa) \in \mathcal{O}(1). \tag{3.10}$$

The inclusion $M_3(\kappa) \in \mathcal{O}(1)$ follows from standard estimates and from the fact that $\frac{1}{\kappa}\big(N_2(\kappa) - N_2(0)\big)$ admits a finite limit as $\kappa \to 0$ (see [7, Lemma 5.1(i)]). Note also that the kernel of $N_2(0)$ is given by

$$N_2(0)(\omega, x, \omega', x') = \frac{1}{4}\sum_{n\in N} f_n(\omega)\, v(\omega, x)\, |x - x'|^2\, v(\omega', x')\, \overline{f_n(\omega')}, \quad (\omega, x), (\omega', x') \in \Omega. \tag{3.11}$$

Now, as already observed, one has $M_1(0) = X + iZ^*Z$, with X, Z bounded self-adjoint operators in $\mathcal{H}$. Therefore it follows that $I_1(0) = S_0 M_1(0) S_0 = S_0 X S_0 + i(ZS_0)^*(ZS_0)$, and one infers from Corollary 2.5 that $ZS_0S_1 = 0$ and $S_1S_0Z^* = 0$. Since $S_1S_0 = S_1 = S_0S_1$, it follows that $ZS_1 = 0$, that $S_1Z^* = 0$, and also that

$$\begin{aligned} S_1 M_1(0)\big(I_0(0)+S_0\big)^{-1} M_1(0) S_1 &= S_1(X + iZ^*Z)\big(I_0(0)+S_0\big)^{-1}(X+iZ^*Z)S_1 \\ &= S_1 X\big(I_0(0)+S_0\big)^{-1} X S_1. \end{aligned}$$

So, this operator is self-adjoint, and thus one infers from (3.10) and (3.11) that $I_2(0) = S_1 M_2(0) S_1$ is the sum of two bounded self-adjoint operators in $S_1\mathcal{H}$.

Since $S_1\mathcal{H}$ is finite-dimensional, 0 is not a limit point of the spectrum of $I_2(0)$. So, the orthogonal projection S_2 on $\ker\big(I_2(0)\big)$ is a finite-rank operator, and Proposition 2.1 applies to $I_2(0) + \kappa\, M_3(\kappa)$. Thus, for $\kappa \in \vartheta(\varepsilon)$ with $\varepsilon > 0$ small enough, the operator $I_3(\kappa) : S_2\mathcal{H} \to S_2\mathcal{H}$ defined by

$$I_3(\kappa) := \sum_{j\geq 0} (-\kappa)^j S_2 \big\{M_3(\kappa)\big(I_2(0)+S_2\big)^{-1}\big\}^{j+1} S_2$$

is uniformly bounded as $\kappa \to 0$. Furthermore, $I_3(\kappa)$ is invertible in $S_2\mathcal{H}$ with bounded inverse satisfying the equation

$$I_2(\kappa)^{-1} = \big(I_2(\kappa)+S_2\big)^{-1} + \frac{1}{\kappa}\big(I_2(\kappa)+S_2\big)^{-1} S_2 I_3(\kappa)^{-1} S_2 \big(I_2(\kappa)+S_2\big)^{-1}.$$

This expression for $I_2(\kappa)^{-1}$ can now be inserted in (3.9) in order to get

$$\begin{aligned} \mathrm{M}(\lambda, \kappa) &= 2\kappa\big(I_0(\kappa)+S_0\big)^{-1} + \big(I_0(\kappa)+S_0\big)^{-1} S_0 \big(I_1(\kappa)+S_1\big)^{-1} S_0 \big(I_0(\kappa)+S_0\big)^{-1} \\ &\quad + \frac{1}{\kappa}\big(I_0(\kappa)+S_0\big)^{-1} S_0 \big(I_1(\kappa)+S_1\big)^{-1} S_1 \big(I_2(\kappa)+S_2\big)^{-1} S_1 \big(I_1(\kappa)+S_1\big)^{-1} S_0 \big(I_0(\kappa)+S_0\big)^{-1} \\ &\quad + \frac{1}{\kappa^2}\big(I_0(\kappa)+S_0\big)^{-1} S_0 \big(I_1(\kappa)+S_1\big)^{-1} S_1 \big(I_2(\kappa)+S_2\big)^{-1} S_2 I_3(\kappa)^{-1} S_2 \big(I_2(\kappa)+S_2\big)^{-1} S_1 \\ &\quad \times \big(I_1(\kappa)+S_1\big)^{-1} S_0 \big(I_0(\kappa)+S_0\big)^{-1}. \end{aligned} \tag{3.12}$$

Fortunately, the iterative procedure stops here. The argument is based on the relation

$$uv(H - \lambda + \kappa^2)^{-1} vu = u - \mathrm{M}(\lambda, \kappa)$$

and the fact that H is a self-adjoint operator. Indeed, if we choose $\kappa = \frac{\varepsilon}{2}(1-i) \in \vartheta(\varepsilon)$, then the inequality $\big\|\kappa^2(H - \lambda + \kappa^2)^{-1}\big\| \leq 1$ holds, and thus

$$\limsup_{\kappa\to 0} \big\|\kappa^2\, \mathrm{M}(\lambda, \kappa)\big\| < \infty. \tag{3.13}$$

So, if we replace $\mathrm{M}(\lambda, \kappa)$ by the expression (3.12) and if we take into account that all factors of the form $\big(I_j(\kappa)+S_j\big)^{-1}$ have a finite limit as $\kappa \to 0$, we infer from (3.13) that

$$\limsup_{\kappa\to 0} \big\|I_3(\kappa)^{-1}\big\|_{\mathscr{B}(S_2\mathcal{H})} < \infty. \tag{3.14}$$

Therefore, it only remains to show that this relation holds not just for $\kappa = \frac{\varepsilon}{2}(1-i)$ but for all $\kappa \in \vartheta(\varepsilon)$. For that purpose, we consider $I_3(\kappa)$ once again, and note that

$$I_3(\kappa) = S_2 M_3(0) S_2 + \kappa M_4(\kappa) \quad \text{with} \quad M_4(\kappa) \in \mathcal{O}(1). \tag{3.15}$$

The precise form of $M_3(0)$ can be computed explicitly, but is irrelevant. Now, since $I_3(0)$ acts in a finite-dimensional space, 0 is an isolated eigenvalue of $I_3(0)$ if $0 \in \sigma\big(I_3(0)\big)$, in which case we write S_3 for the corresponding Riesz projection. Then, the operator $I_3(0) + S_3$ is invertible with bounded inverse, and (3.15) implies that $I_3(\kappa) + S_3$ is also invertible with bounded inverse for $\kappa \in \vartheta(\varepsilon)$ with $\varepsilon > 0$ small enough. In addition, one has

$$\big(I_3(\kappa) + S_3\big)^{-1} = \big(I_3(0) + S_3\big)^{-1} + \mathcal{O}(\kappa).$$

By the inversion formula given in [7, Lemma 2.1], one infers that $S_3 - S_3\big(I_3(\kappa) + S_3\big)^{-1} S_3$ is invertible in $S_3\mathcal{H}$ with bounded inverse and that the following equalities hold

$$\begin{aligned} I_3(\kappa)^{-1} &= \big(I_3(\kappa) + S_3\big)^{-1} + \big(I_3(\kappa) + S_3\big)^{-1} S_3 \big\{S_3 - S_3\big(I_3(\kappa) + S_3\big)^{-1} S_3\big\}^{-1} S_3 \big(I_3(\kappa) + S_3\big)^{-1} \\ &= \big(I_3(\kappa) + S_3\big)^{-1} + \big(I_3(\kappa) + S_3\big)^{-1} S_3 \big\{S_3 - S_3\big(I_3(0) + S_3\big)^{-1} S_3 + \mathcal{O}(\kappa)\big\}^{-1} S_3 \big(I_3(\kappa) + S_3\big)^{-1}. \end{aligned}$$

This implies that (3.14) holds for some $\kappa \in \vartheta(\varepsilon)$ if and only if the operator $S_3 - S_3\big(I_3(0) + S_3\big)^{-1} S_3$ is invertible in $S_3\mathcal{H}$ with bounded inverse. But, we already know from what precedes that (3.14) holds for $\kappa = \frac{\varepsilon}{2}(1-i)$. So, the operator $S_3 - S_3\big(I_3(0) + S_3\big)^{-1} S_3$ is invertible in $S_3\mathcal{H}$ with bounded inverse, and thus (3.14) holds for arbitrary $\kappa \in \vartheta(\varepsilon)$.

(ii) Assume now that $\lambda \in \sigma_{\rm p}(H) \setminus \tau$ and set $J_0(\kappa) := T_0 + \kappa^2 T_1(\kappa)$ with

$$T_0 := u + \sum_n v\big(\mathcal{P}_n \otimes R^0(\lambda - \lambda_n)\big)\, v$$

and

$$T_1(\kappa) := \frac{1}{\kappa^2} \sum_n v\left\{\mathcal{P}_n \otimes \big(R^0(\lambda - \kappa^2 - \lambda_n) - R^0(\lambda - \lambda_n)\big)\right\} v.$$

Then, one infers from Lemma 3.1(b) that $\|T_1(\kappa)\|$ is uniformly bounded as $\kappa \to 0$. Also, the assumptions of Corollary 2.9 hold for the operator T_0, the Riesz projection S associated with the value $0 \in \sigma(T_0)$ is an orthogonal projection, and Proposition 2.1 applies for $J_0(\kappa)$. It follows that for $\kappa \in \vartheta(\varepsilon)$ with $\varepsilon > 0$ small enough, the operator $J_1(\kappa) : S\mathcal{H} \to S\mathcal{H}$ defined by

$$J_1(\kappa) := \sum_{j \geq 0} (-\kappa^2)^j S \big\{T_1(\kappa)(T_0 + S)^{-1}\big\}^{j+1} S$$

is uniformly bounded as $\kappa \to 0$. Furthermore, $J_1(\kappa)$ is invertible in $S\mathcal{H}$ with bounded inverse satisfying the equation

$$\mathrm{M}(\lambda, \kappa) = \big(J_0(\kappa) + S\big)^{-1} + \frac{1}{\kappa^2}\big(J_0(\kappa) + S\big)^{-1} S J_1(\kappa)^{-1} S \big(J_0(\kappa) + S\big)^{-1}. \tag{3.16}$$

Fortunately, the iterative procedure already stops here. Indeed, the argument is similar to the one presented above once we observe that

$$J_1(\kappa) = S T_1(0) S + \kappa T_2(\kappa) \quad \text{with} \quad T_2(\kappa) \in \mathcal{O}(1).$$

□

Corollary 3.3. *Suppose that $V \in \mathrm{L}^\infty(\Omega; \mathbb{R})$ has bounded support. Then, the point spectrum of H has no accumulation point (except possibly at $+\infty$).*

Proof. We already know that the eigenvalues of H in $\sigma(H)\setminus\tau$ are of finite multiplicity and can accumulate at points of τ only (see [11, Thm. 3.4(b)]). Thus, it is sufficient to show that there is no accumulation of eigenvalues at points of τ. To show this, suppose by absurd that there is an accumulation of eigenvalues at some point $\lambda\in\tau$. Then, the validity of the expansion (3.16) for each eigenvalue of the corresponding cluster of eigenvalues contradicts the validity of the expansion (3.12) at the point λ. Thus, there is no accumulation of eigenvalues at points of τ, and the claim is proved. □

We end up this section with some auxiliary results which will be useful later on. All notations and definitions are borrowed from the proof of Proposition 3.2. The only change is that we extend by 0 the operators defined originally on subspaces of $\mathcal{H}$ to get operators defined on all of $\mathcal{H}$.

Lemma 3.4. *Take $j,k\in\{0,1,2\}$ with $j\ge k$ and $\kappa\in\vartheta(\varepsilon)$ with $\varepsilon>0$ small enough. Then, one has in $\mathscr{B}(\mathcal{H})$*

$$\big[S_j,\big(I_k(\kappa)+S_k\big)^{-1}\big]\in\mathcal{O}(\kappa).$$

Proof. The fact that S_j is the orthogonal projection on the kernel of $I_j(0)$ and the relations $S_kS_j=S_j=S_jS_k$ imply that $[S_k,S_j]=0$ and $[I_k(0),S_j]=0$. Thus, one has the equalities

$$\begin{aligned}\big[S_j,\big(I_k(\kappa)+S_k\big)^{-1}\big]&=\big(I_k(\kappa)+S_k\big)^{-1}\big[I_k(\kappa)+S_k,S_j\big]\big(I_k(\kappa)+S_k\big)^{-1}\\&=\big(I_k(\kappa)+S_k\big)^{-1}\big[I_k(0)+\mathcal{O}(\kappa)+S_k,S_j\big]\big(I_k(\kappa)+S_k\big)^{-1}\\&=\big(I_k(\kappa)+S_k\big)^{-1}\big[\mathcal{O}(\kappa),S_j\big]\big(I_k(\kappa)+S_k\big)^{-1},\end{aligned}$$

which implies the claim. □

Given $\lambda\in\tau$, we recall that $N=\big\{n\ge1\mid\lambda_n=\lambda\big\}$ and $\mathcal{P}=\sum_{n\in N}\mathcal{P}_n$.

Lemma 3.5. *Let $\lambda\in\tau$ and let $\mathcal{G}$ be an auxiliary Hilbert space.*

(a) For each $n\in N$, one has $(\mathcal{P}_n\otimes1)\,vS_0=0$.

(b) For each $n\notin N$ and $B_n\in\mathscr{B}(\mathcal{H},\mathcal{G})$ such that $B_n^*B_n=\operatorname{Im}\big\{v\big(\mathcal{P}_n\otimes R^0(\lambda-\lambda_n)\big)v\big\}$, one has $S_1B_n^*=0$ and $B_nS_1=0$.

Proof. The first claim follows from the fact that S_0 is the orthogonal projection on $\ker\big(v(\mathcal{P}\otimes1)\,v\big)$. The second claim follows from Lemma 2.5 applied with $Z_n=B_nS_0$ and

$$A_0=S_0M_1(0)S_0=S_0\left\{N_1(0)+u+\sum_{n\notin N}v\big(\mathcal{P}_n\otimes R^0(\lambda-\lambda_n)\big)v\right\}S_0$$

if one takes into account the relations $S_0S_1=S_1=S_1S_0$. □

For what follows, we recall that Q is the multiplication operator by the variable in $\mathsf{L}^2(\mathbb{R})$.

Lemma 3.6. *One has*

(a) $\big(I_0(0)+S_0\big)^{-1}XS_2=0$, with X the real part of the operator $M_1(0)$,

(b) $S_2(1\otimes Q)\,v(f_n\otimes1)=0$ for all $n\in N$.

Proof. First, we recall from the proof of Proposition 3.2 that

$$I_2(0)=S_1M_2(0)S_1=S_1N_2(0)S_1-2S_1X\big(I_0(0)+S_0\big)^{-1}XS_1,$$

with $N_2(0)$ given (in the usual bra-ket notation) by

$$N_2(0) = \frac{1}{4}\sum_{n\in N}\big\{\big|(1\otimes Q^2)\,v(f_n\otimes 1)\big\rangle\big\langle v(f_n\otimes 1)\big| + \big|v(f_n\otimes 1)\big\rangle\big\langle(1\otimes Q^2)\,v(f_n\otimes 1)\big|$$
$$- 2\big|(1\otimes Q)\,v(f_n\otimes 1)\big\rangle\big\langle(1\otimes Q)\,v(f_n\otimes 1)\big|\big\}.$$

Now, let $\varphi \in S_2\mathcal{H}$. Then, we have $I_2(0)\varphi = 0$ and

$$\big\langle\varphi, N_2(0)\varphi\big\rangle = 2\big\langle\varphi, X\big(I_0(0)+S_0\big)^{-1}X\varphi\big\rangle. \tag{3.17}$$

In addition, one infers from the relation $S_2 = S_0S_2$ and Lemma 3.5(a) that

$$\big\langle\varphi, \big\{(1\otimes Q^2)\,v(f_n\otimes 1)\big\rangle\big\langle v(f_n\otimes 1)\big|\big\}\varphi\big\rangle = \big\langle\varphi, (1\otimes Q^2)v(f_n\otimes 1)\big\rangle\big\langle S_0\,v(f_n\otimes 1), \varphi\big\rangle = 0,$$

and thus (3.17) reduces to

$$-\left\langle\varphi, \sum_{n\in N}\big\{\big|(1\otimes Q)\,v(f_n\otimes 1)\big\rangle\big\langle(1\otimes Q)\,v(f_n\otimes 1)\big|\big\}\varphi\right\rangle = 4\big\langle\varphi, X\big(I_0(0)+S_0\big)^{-1}X\varphi\big\rangle.$$

Since both operators are positive, both sides of the equality are equal to 0, from which the claims are easily deduced. $\square$

3.2 Scattering theory and spectral representation

In this section, we recall some basics on the scattering theory for the pair $\{H_0, H\}$ and on the spectral decomposition for H_0. Our assumptions on V are the ones stated in Proposition 3.2.

Under our assumptions, it is a well-known that the wave operators

$$W_\pm := \text{s-}\lim_{t\to\pm\infty} e^{itH}\,e^{-itH_0}$$

exist and are complete (see [11, Cor. 3.5(b)]). As a consequence, the scattering operator $S := W_+^*W_-$ is a unitary operator in $\mathcal{H}$ which commutes with H_0, and thus S is decomposable in the spectral representation of H_0. So, in order to proceed, we start by recalling the spectral representation of H_0. For that purpose, we define for each $\lambda \in [\lambda_1, \infty)$ the finite set

$$\mathbb{N}(\lambda) := \big\{n \geq 1 \mid \lambda_n \leq \lambda\big\}$$

and the finite-dimensional space

$$\mathscr{H}(\lambda) := \bigoplus_{n\in\mathbb{N}(\lambda)}\big\{\mathcal{P}_n\,\mathsf{L}^2(\Sigma)\oplus\mathcal{P}_n\,\mathsf{L}^2(\Sigma)\big\},$$

with λ_n and $\mathcal{P}_n$ as in Section 3. Note that $\mathscr{H}(\lambda)$ is naturally embedded in $\mathscr{H}(\infty) := \bigoplus_{n\geq 1}\big\{\mathcal{P}_n\,\mathsf{L}^2(\Sigma)\oplus \mathcal{P}_n\,\mathsf{L}^2(\Sigma)\big\}$. Now, for any $\xi\in\mathbb{R}$, we let $\gamma(\xi) : \mathscr{S}(\mathbb{R}) \to \mathbb{C}$ be the trace operator given by $\gamma(\xi)f = f(\xi)$, with $\mathscr{S}(\mathbb{R})$ the Schwartz space on $\mathbb{R}$. Also, we define for each $\lambda \in [\lambda_1,\infty)\setminus\tau$ the operator $T(\lambda) : \mathsf{L}^2(\Sigma)\odot\mathscr{S}(\mathbb{R}) \to \mathscr{H}(\lambda)$ by

$$\big(T(\lambda)\,\varphi\big)_n := (\lambda-\lambda_n)^{-1/4}\big\{\big(\mathcal{P}_n\otimes\gamma(-\sqrt{\lambda-\lambda_n})\big)\varphi, \big(\mathcal{P}_n\otimes\gamma(\sqrt{\lambda-\lambda_n})\big)\varphi\big\}, \quad n\geq 1.$$

Some regularity properties of the map $\lambda\mapsto T(\lambda)$ have been established in [11, Lemma 2.4], and additional properties are derived below for the related map $\lambda \mapsto \mathscr{F}_0(\lambda)$ which we now define.

Let $\mathscr{F} : \mathsf{L}^2(\mathbb{R}) \to \mathsf{L}^2(\mathbb{R})$ be the Fourier transform and let $\mathscr{H} := \int_{[\lambda_1,\infty)}^{\oplus}\mathscr{H}(\lambda)\,\mathrm{d}\lambda$. Then, the operator $\mathscr{F}_0 : \mathcal{H}\to\mathscr{H}$ given by

$$(\mathscr{F}_0\,\varphi)(\lambda) \equiv \mathscr{F}_0(\lambda)\,\varphi := 2^{-1/2}\,T(\lambda)(1\otimes\mathscr{F})\,\varphi, \quad \lambda\in[\lambda_1,\infty)\setminus\tau,\ \varphi\in\mathsf{L}^2(\Sigma)\odot\mathscr{S}(\mathbb{R}),$$

is unitary and satisfies $\mathscr{F}_0 H_0 \mathscr{F}_0^* = \int_{[\lambda_1,\infty)}^{\oplus} \lambda \, d\lambda$ (see [11, Prop. 2.5]). We shall need some asymptotic expansions for the map $\lambda \mapsto \mathscr{F}_0(\lambda)$ in neighbourhoods of points $\lambda \in \tau \cup \sigma_p(H)$. For this, we define for each $\lambda > \lambda_1$, each $n \geq 1$ such that $\lambda_n < \lambda$, and each $\sigma \in \{+,-\}$

$$\mathscr{F}_0(\lambda; n, \sigma)\,\varphi := 2^{-1/2}(\lambda - \lambda_n)^{-1/4}\big(\mathcal{P}_n \otimes \gamma(\sigma\sqrt{\lambda - \lambda_n})\mathscr{F}\big)\varphi, \quad \varphi \in L^2(\Sigma) \odot \mathscr{S}(\mathbb{R}).$$

The operator $\mathscr{F}_0(\lambda; n, \sigma) : L^2(\Sigma) \odot \mathscr{S}(\mathbb{R}) \to \mathcal{P}_n L^2(\Sigma)$ is defined on a slightly larger set of λ than the operator $\mathscr{F}_0(\lambda) : L^2(\Sigma) \odot \mathscr{S}(\mathbb{R}) \to \mathscr{H}(\lambda)$. Also, we define (similarly to the sets $\vartheta(\varepsilon)$ of the previous section) the sets

$$\partial\vartheta(\varepsilon) := \big\{\kappa \in \mathbb{C} \mid \kappa \in (0, \varepsilon) \cup (0, -i\varepsilon)\big\}, \quad \varepsilon > 0,$$

for which $-\kappa^2 \in (-\varepsilon^2, \varepsilon^2) \setminus \{0\}$ if $\kappa \in \partial\vartheta(\varepsilon)$, and we let $L^2_s(\mathbb{R})$ be the domain of $\langle Q\rangle^s$, $s \in \mathbb{R}$, endowed with the graph norm. Then, given $\lambda \in \tau \cup \sigma_p(H)$, we consider for each $\kappa \in \partial\vartheta(\varepsilon)$ with $\varepsilon > 0$ small enough the asymptotic expansion in κ of the operator $\mathscr{F}_0(\lambda - \kappa^2; n, \sigma)$. If $\lambda_n < \lambda$, one has for $\kappa \in \partial\vartheta(\varepsilon)$ with $\varepsilon > 0$ small enough

$$(\lambda - \kappa^2 - \lambda_n)^{-1/4} = (\lambda - \lambda_n)^{-1/4}\left(1 + \frac{\kappa^2}{4(\lambda - \lambda_n)} + \mathcal{O}(\kappa^4)\right).$$

Similarly, if $s > 0$ is big enough and if $\sigma \in \{+,-\}$, one has in $\mathscr{B}\big(L^2_s(\mathbb{R}), \mathbb{C}\big)$

$$\gamma(\sigma\sqrt{\lambda - \kappa^2 - \lambda_n})\mathscr{F} = \gamma(\sigma\sqrt{\lambda - \lambda_n})\mathscr{F}\left(1 + \frac{i\sigma\kappa^2}{2\sqrt{\lambda - \lambda_n}}\,Q\right) + \mathcal{O}(\kappa^4).$$

As a consequence, we have in $\mathscr{B}\big(L^2(\Sigma) \otimes L^2_s(\mathbb{R}); \mathcal{P}_n L^2(\Sigma)\big)$

$$\mathscr{F}_0(\lambda - \kappa^2; n, \sigma) = \mathscr{F}_0(\lambda; n, \sigma)\left(1 + \frac{\kappa^2}{4(\lambda - \lambda_n)} + \frac{i\sigma\kappa^2}{2\sqrt{\lambda - \lambda_n}}\,Q\right) + \mathcal{O}(\kappa^4). \tag{3.18}$$

On the other hand, if $\lambda = \lambda_n \in \tau$ and $-\kappa^2 > 0$ (that is, $i\kappa > 0$), then one obtains in $\mathscr{B}\big(L^2(\Sigma) \otimes L^2_s(\mathbb{R}), \mathcal{P}_n L^2(\Sigma)\big)$

$$\mathscr{F}_0(\lambda - \kappa^2; n, \sigma) = (-\kappa^2)^{-1/4}\gamma_0(n) - i\sigma(-\kappa^2)^{1/4}\gamma_1(n) + \mathcal{O}(|\kappa|^{3/2}) \tag{3.19}$$

with $\gamma_j(n) : L^2(\Sigma) \otimes L^2_s(\mathbb{R}) \to \mathcal{P}_n L^2(\Sigma)$ the operator given by

$$\big(\gamma_j(n)\varphi\big)(\omega) := \frac{1}{2j!\sqrt{\pi}} \int_{\mathbb{R}} x^j \big((\mathcal{P}_n \otimes 1)\varphi\big)(\omega, x)\, dx \quad \text{for almost every } \omega \in \Sigma.$$

With these expansions at hand, we can start the study of the regularity properties of the scattering matrix at thresholds or at embedded eigenvalues. Before that, we just need to give a final auxiliary result. Recall that the orthogonal projections S_0 and S_1 have been introduced in the proof of Proposition 3.2.

Lemma 3.7. *Take $\lambda \in \tau$, $\sigma \in \{+,-\}$, and $\kappa \in \partial\vartheta(\varepsilon)$ with $\varepsilon > 0$ small enough.*

(a) For $n \geq 1$ such that $\lambda_n < \lambda$, one has $\mathscr{F}_0(\lambda - \kappa^2; n, \sigma)\, vS_1 \in \mathcal{O}(\kappa^2)$.

(b) For $n \geq 1$ such that $\lambda_n = \lambda$ and for $-\kappa^2 > 0$, one has $\mathscr{F}_0(\lambda - \kappa^2; n, \sigma)\, vS_0 = 0$.

Proof. (a) Due to the expansion (3.18), it is sufficient to show the equality $\mathscr{F}_0(\lambda; n, \sigma)vS_1 = 0$. For that purpose, we define the operator $B_n : \mathcal{H} \to \mathcal{P}_n L^2(\Sigma) \oplus \mathcal{P}_n L^2(\Sigma)$ by

$$B_n\varphi := \pi^{1/2}\big\{\mathscr{F}_0(\lambda; n, -)\, v\,\varphi, \mathscr{F}_0(\lambda; n, +)\, v\,\varphi\big\},$$

and note that $B_n^* B_n = \operatorname{Im}\big\{v\big(\mathcal{P}_n \otimes R^0(\lambda - \lambda_n)\big)v\big\}$. The mentioned equality then follows from Lemma 3.5(b).

(b) The claim is a direct consequence of the identity

$$\mathscr{F}_0(\lambda - \kappa^2; n, \sigma)\, vS_0 = \mathscr{F}_0(\lambda - \kappa^2; n, \sigma)(\mathcal{P}_n \otimes 1)\, vS_0$$

and Lemma 3.5(a). $\square$

3.3 Continuity of the scattering matrix

Since the scattering operator S commutes with H_0, it follows from the spectral decomposition of H_0 that

$$\mathscr{F}_0\, S\, \mathscr{F}_0^* = \int_{[\lambda_1,\infty)}^{\oplus} S(\lambda)\,\mathrm{d}\lambda,$$

where $S(\lambda)$, the scattering matrix at energy λ, is defined and is a unitary operator in $\mathscr{H}(\lambda)$ for almost every $\lambda \in [\lambda_1, \infty)$. In addition, one can obtain a convenient stationary formula for $S(\lambda)$ using time-dependent scattering theory. For instance, if one uses the results of [11, Sec. 3.1] and relation (3.3), one obtains for each $\lambda \in [\lambda_1, \infty) \setminus \{\tau \cup \sigma_{\mathrm{p}}(H)\}$ the equality in $\mathscr{B}\big(\mathscr{H}(\lambda)\big)$

$$S(\lambda) = 1 - 2\pi i\, \mathscr{F}_0(\lambda)\, v\big(u + vR_0(\lambda)\, v\big)^{-1} v\, \mathscr{F}_0(\lambda)^*,$$

and that the map

$$[\lambda_1, \infty) \setminus \{\tau \cup \sigma_{\mathrm{p}}(H)\} \ni \lambda \mapsto S(\lambda) \in \mathscr{H}(\infty)$$

is a k-times continuously differentiable, for any $k \geq 0$.

Since the regularity of the map $\lambda \mapsto S(\lambda)$ is already known when $\lambda \in [\lambda_1, \infty) \setminus \{\tau \cup \sigma_{\mathrm{p}}(H)\}$, we now describe the behavior of $S(\lambda)$ as λ approaches points of $\tau \cup \sigma_{\mathrm{p}}(H)$. To do this, we decompose the scattering matrix $S(\lambda)$ into a collection of channel scattering matrices corresponding to the transverse modes of the waveguide. Namely, for $\lambda \in [\lambda_1, \infty) \setminus \{\tau \cup \sigma_{\mathrm{p}}(H)\}$, for $n, n' \geq 1$ such that $\lambda_n < \lambda$ and $\lambda_{n'} < \lambda$, and for $\sigma, \sigma' \in \{+, -\}$ we define the operators $S(\lambda; n, \sigma, n', \sigma') \in \mathscr{B}\big(\mathcal{P}_{n'}\, \mathrm{L}^2(\Sigma), \mathcal{P}_n\, \mathrm{L}^2(\Sigma)\big)$ by

$$S(\lambda; n, \sigma, n', \sigma') := \delta_{n\sigma n'\sigma'} - 2\pi i\, \mathscr{F}_0(\lambda; n, \sigma)\, v\big(u + vR_0(\lambda)\, v\big)^{-1} v\, \mathscr{F}_0(\lambda; n', \sigma')^*$$

with $\delta_{n\sigma n'\sigma'} := 1$ if $(n, \sigma) = (n', \sigma')$, and $\delta_{n\sigma n'\sigma'} := 0$ otherwise.

We consider separately the continuity at thresholds and the continuity at embedded eigenvalues, starting with the thresholds. Note that for each $\lambda \in \tau$, a channel can either be already open (in which case one has to show the existence and the equality of the limits from the right and from the left), or can open at the energy λ (in which case one has only to show the existence of the limit from the right).

Proposition 3.8. *Suppose that $V \in \mathrm{L}^\infty(\Omega; \mathbb{R})$ has bounded support, take $\lambda \in \tau$, $\kappa \in \partial\vartheta(\varepsilon)$ with $\varepsilon > 0$ small enough, $n, n' \geq 1$, and $\sigma, \sigma' \in \{+, -\}$.*

(a) If $\lambda_n < \lambda$ and $\lambda_{n'} < \lambda$, then the limit $\lim_{\kappa \to 0} S(\lambda - \kappa^2; n, \sigma, n', \sigma')$ exists.

(b) If $\lambda_n \leq \lambda$, $\lambda_{n'} \leq \lambda$ and $-\kappa^2 > 0$, then the limit $\lim_{\kappa \to 0} S(\lambda - \kappa^2; n, \sigma, n', \sigma')$ exists.

Before giving the proof, we define for $2 \geq j \geq k \geq 0$

$$C_{jk}(\kappa) := \big[S_j, \big(I_k(\kappa) + S_k\big)^{-1}\big] \in \mathcal{O}(\kappa) \qquad \text{and} \qquad C'_{jk}(0) := \lim_{\kappa \to 0} \frac{1}{\kappa}\, C_{jk}(\kappa),$$

and observe that (3.12) can be rewritten as

$$
\begin{aligned}
&\mathrm{M}(\lambda,\kappa)\\
&= 2\kappa\big(I_0(\kappa)+S_0\big)^{-1}\\
&\quad+\Big(S_0\big(I_0(\kappa)+S_0\big)^{-1}-C_{00}(\kappa)\Big)S_0\big(I_1(\kappa)+S_1\big)^{-1}S_0\Big(\big(I_0(\kappa)+S_0\big)^{-1}S_0+C_{00}(\kappa)\Big)\\
&\quad+\frac{1}{\kappa}\big(I_0(\kappa)+S_0\big)^{-1}\Big(S_1\big(I_1(\kappa)+S_1\big)^{-1}-S_0C_{11}(\kappa)\Big)S_1\big(I_2(\kappa)+S_2\big)^{-1}S_1\\
&\qquad\times\Big(\big(I_1(\kappa)+S_1\big)^{-1}S_1+C_{11}(\kappa)S_0\Big)\big(I_0(\kappa)+S_0\big)^{-1}\\
&\quad+\frac{1}{\kappa^2}\big(I_0(\kappa)+S_0\big)^{-1}S_0\big(I_1(\kappa)+S_1\big)^{-1}\Big(S_2\big(I_2(\kappa)+S_2\big)^{-1}-S_1C_{22}(\kappa)\Big)S_2I_3(\kappa)^{-1}S_2\\
&\qquad\times\Big(\big(I_2(\kappa)+S_2\big)^{-1}S_2+C_{22}(\kappa)S_1\Big)\big(I_1(\kappa)+S_1\big)^{-1}S_0\big(I_0(\kappa)+S_0\big)^{-1}\\
&= 2\kappa\big(I_0(\kappa)+S_0\big)^{-1}\\
&\quad+\Big(S_0\big(I_0(\kappa)+S_0\big)^{-1}-C_{00}(\kappa)\Big)S_0\big(I_1(\kappa)+S_1\big)^{-1}S_0\Big(\big(I_0(\kappa)+S_0\big)^{-1}S_0+C_{00}(\kappa)\Big)\\
&\quad+\frac{1}{\kappa}\bigg\{\Big(S_1\big(I_0(\kappa)+S_0\big)^{-1}-C_{10}(\kappa)\Big)\big(I_1(\kappa)+S_1\big)^{-1}-\Big(S_0\big(I_0(\kappa)+S_0\big)^{-1}-C_{00}(\kappa)\Big)C_{11}(\kappa)\bigg\}\\
&\qquad\times S_1\big(I_2(\kappa)+S_2\big)^{-1}S_1\bigg\{\big(I_1(\kappa)+S_1\big)^{-1}\Big(\big(I_0(\kappa)+S_0\big)^{-1}S_1+C_{10}(\kappa)\Big)\\
&\qquad+C_{11}(\kappa)\Big(\big(I_0(\kappa)+S_0\big)^{-1}S_0+C_{00}(\kappa)\Big)\bigg\}\\
&\quad+\frac{1}{\kappa^2}\bigg\{\bigg[\Big(S_2\big(I_0(\kappa)+S_0\big)^{-1}-C_{20}(\kappa)\Big)\big(I_1(\kappa)+S_1\big)^{-1}\\
&\qquad-\Big(S_0\big(I_0(\kappa)+S_0\big)^{-1}-C_{00}(\kappa)\Big)C_{21}(\kappa)\bigg]\big(I_2(\kappa)+S_2\big)^{-1}\\
&\qquad-\bigg[\Big(S_1\big(I_0(\kappa)+S_0\big)^{-1}-C_{10}(\kappa)\Big)\big(I_1(\kappa)+S_1\big)^{-1}\\
&\qquad-\Big(S_0\big(I_0(\kappa)+S_0\big)^{-1}-C_{00}(\kappa)\Big)C_{11}(\kappa)\bigg]C_{22}(\kappa)\bigg\}S_2I_3(\kappa)^{-1}S_2\\
&\qquad\times\bigg\{\big(I_2(\kappa)+S_2\big)^{-1}\bigg[\big(I_1(\kappa)+S_1\big)^{-1}\Big(\big(I_0(\kappa)+S_0\big)^{-1}S_2+C_{20}(\kappa)\Big)\\
&\qquad+C_{21}(\kappa)\Big(\big(I_0(\kappa)+S_0\big)^{-1}S_0+C_{00}(\kappa)\Big)\bigg]\\
&\qquad+C_{22}(\kappa)\bigg[\big(I_1(\kappa)+S_1\big)^{-1}\Big(\big(I_0(\kappa)+S_0\big)^{-1}S_1+C_{10}(\kappa)\Big)\\
&\qquad+C_{11}(\kappa)\Big(\big(I_0(\kappa)+S_0\big)^{-1}S_0+C_{00}(\kappa)\Big)\bigg]\bigg\}.
\end{aligned}
$$

The interest in this formulation is that the projections S_j (which lead to simplifications in the proof) have been put into evidence at the beginning or at the end of each term.

Proof. (a) Some lengthy, but direct, computations taking into account the above expansion for $\mathrm{M}(\lambda,\kappa)$, the relation $\big(I_j(0)+S_j\big)^{-1}S_j=S_j$, the expansion (3.18) for $\mathscr{F}_0(\lambda-\kappa^2;n,\sigma)$ and $\mathscr{F}_0(\lambda-\kappa^2;n',\sigma')$ and

Lemma 3.7(a) lead to the equality

$$\begin{aligned}&\lim_{\kappa\to 0}\mathscr{F}_0(\lambda-\kappa^2;n,\sigma)\,v\,\mathrm{M}(\lambda,\kappa)\,v\,\mathscr{F}_0(\lambda-\kappa^2;n',\sigma')^*\\&=\mathscr{F}_0(\lambda;n,\sigma)\,vS_0\big(I_1(0)+S_1\big)^{-1}S_0v\,\mathscr{F}_0(\lambda;n',\sigma')^*\\&\quad-\mathscr{F}_0(\lambda;n,\sigma)\,v\big(C_{20}'(0)+S_0C_{21}'(0)\big)S_2I_3(0)^{-1}S_2\big(C_{20}'(0)+C_{21}'(0)S_0\big)v\,\mathscr{F}_0(\lambda;n',\sigma')^*.\end{aligned}$$

Since

$$S(\lambda-\kappa^2;n,\sigma,n',\sigma')-\delta_{n\sigma n'\sigma'}=-2\pi i\mathscr{F}_0(\lambda-\kappa^2;n,\sigma)\,v\mathrm{M}(\lambda,\kappa)\,v\mathscr{F}_0(\lambda-\kappa^2;n',\sigma')^*,\tag{3.20}$$

this proves the claim.

(b.1) We first consider the case $\lambda_n<\lambda$, $\lambda_{n'}=\lambda$ and $-\kappa^2>0$ (the case $\lambda_n=\lambda$, $\lambda_{n'}<\lambda$ and $-\kappa^2>0$ is not presented since it is similar). A direct inspection taking into account the above expansion for $\mathrm{M}(\lambda,\kappa)$, the relation $\big(I_j(\kappa)+S_j\big)^{-1}=\big(I_j(0)+S_j\big)^{-1}+\mathcal{O}(\kappa)$ and the relation $\big(I_j(0)+S_j\big)^{-1}S_j=S_j$ leads to the equation

$$\begin{aligned}&\mathscr{F}_0(\lambda-\kappa^2;n,\sigma)\,v\,\mathrm{M}(\lambda,\kappa)\,v\,\mathscr{F}_0(\lambda-\kappa^2;n',\sigma')^*\\&=\mathscr{F}_0(\lambda-\kappa^2;n,\sigma)\,v\,\bigg\{\mathcal{O}(\kappa)+S_0\big(I_1(\kappa)+S_1\big)^{-1}S_0+\frac{1}{\kappa}\big(S_1+\mathcal{O}(\kappa)\big)S_1\big(I_2(\kappa)+S_2\big)^{-1}S_1\big(S_1+\mathcal{O}(\kappa)\big)\\&\quad+\frac{1}{\kappa^2}\Big[\mathcal{O}(\kappa^2)+S_2\big(I_0(\kappa)+S_0\big)^{-1}\big(I_1(\kappa)+S_1\big)^{-1}\big(I_2(\kappa)+S_2\big)^{-1}-C_{20}(\kappa)-S_0C_{21}(\kappa)\\&\quad-S_1C_{22}(\kappa)\Big]S_2I_3(\kappa)^{-1}S_2\Big[\mathcal{O}(\kappa^2)+\big(I_2(\kappa)+S_2\big)^{-1}\big(I_1(\kappa)+S_1\big)^{-1}\big(I_0(\kappa)+S_0\big)^{-1}S_2\\&\quad+C_{20}(\kappa)+C_{21}(\kappa)S_0+C_{22}(\kappa)S_1\Big]\bigg\}\,v\,\mathscr{F}_0(\lambda-\kappa^2;n',\sigma')^*.\end{aligned}\tag{3.21}$$

Applying Lemma 3.7 to the previous equation gives

$$\begin{aligned}&\mathscr{F}_0(\lambda-\kappa^2;n,\sigma)\,v\,\mathrm{M}(\lambda,\kappa)\,v\,\mathscr{F}_0(\lambda-\kappa^2;n',\sigma')^*\\&=\mathscr{F}_0(\lambda-\kappa^2;n,\sigma)\,v\,\bigg\{\mathcal{O}(\kappa)-\frac{1}{\kappa^2}\Big[\mathcal{O}(\kappa^2)+C_{20}(\kappa)+S_0C_{21}(\kappa)\Big]S_2I_3(\kappa)^{-1}S_2\\&\qquad\times\Big[\mathcal{O}(\kappa^2)+C_{20}(\kappa)\Big]\bigg\}\,v\,\mathscr{F}_0(\lambda-\kappa^2;n',\sigma')^*.\end{aligned}$$

Finally, taking into account the expansion (3.18) for $\mathscr{F}_0(\lambda-\kappa^2;n,\sigma)$ and the expansion (3.19) for $\mathscr{F}_0(\lambda-\kappa^2;n',\sigma')$, one ends up with

$$\begin{aligned}&\mathscr{F}_0(\lambda-\kappa^2;n,\sigma)\,v\,\mathrm{M}(\lambda,\kappa)\,v\,\mathscr{F}_0(\lambda-\kappa^2;n',\sigma')^*\\&=(-\kappa^2)^{-5/4}\mathscr{F}_0(\lambda;n,\sigma)\,v\,\Big[\mathcal{O}(\kappa^2)+C_{20}(\kappa)+S_0C_{21}(\kappa)\Big]S_2I_3(\kappa)^{-1}S_2\Big[\mathcal{O}(\kappa^2)+C_{20}(\kappa)\Big]\,v\,\gamma_0(n')^*\\&\quad+\mathcal{O}(|\kappa|^{1/2}),\end{aligned}\tag{3.22}$$

where $\gamma_0(n')^*$ is given by $\gamma_0(n')^*\psi=\frac{1}{2\sqrt{\pi}}\,\psi\otimes 1$ for any $\psi\in\mathcal{P}_{n'}\,\mathrm{L}^2(\Sigma)$.

Now, we have that

$$C_{20}(\kappa)=2\kappa\,\big(I_0(0)+S_0\big)^{-1}\big[M_1(0),S_2\big]\big(I_0(0)+S_0\big)^{-1}+\mathcal{O}(\kappa^2).$$

Therefore, the relation $S_2\big(I_0(0)+S_0\big)^{-1}=S_2$, the equality $S_2M_1(0)=S_2X$ and Lemma 3.6(a) imply that

$$\begin{aligned}S_2C_{20}(\kappa)\,v\,\gamma_0(n')^*&=2\kappa\,S_2\big(I_0(0)+S_0\big)^{-1}\big[M_1(0),S_2\big]\big(I_0(0)+S_0\big)^{-1}v(\mathcal{P}_{n'}\otimes 1)\gamma_0(n')^*+\mathcal{O}(\kappa^2)\\&=2\kappa\big(S_2M_1(0)S_2-S_2M_1(0)\big)\big(I_0(0)+S_0\big)^{-1}v(\mathcal{P}_{n'}\otimes 1)\gamma_0(n')^*+\mathcal{O}(\kappa^2)\\&=-2\kappa\,S_2X\big(I_0(0)+S_0\big)^{-1}v(\mathcal{P}_{n'}\otimes 1)\gamma_0(n')^*+\mathcal{O}(\kappa^2)\\&=\mathcal{O}(\kappa^2).\end{aligned}$$

In consequence, one infers from (3.22) that $\mathscr{F}_0(\lambda-\kappa^2;n,\sigma)\,v\,\mathrm{M}(\lambda,\kappa)\,v\,\mathscr{F}_0(\lambda-\kappa^2;n',\sigma')^*$ vanishes as $\kappa\to 0$, and thus that the limit $\lim_{\kappa\to 0} S(\lambda-\kappa^2;n,\sigma,n',\sigma')$ also vanishes by (3.20).

(b.2) We are left with the case $\lambda_n=\lambda=\lambda_{n'}$ and $-\kappa^2>0$. An application of Lemma 3.7(b) to (3.21) gives

$$\begin{aligned}
&\mathscr{F}_0(\lambda-\kappa^2;n,\sigma)\,v\,\mathrm{M}(\lambda,\kappa)\,v\,\mathscr{F}_0(\lambda-\kappa^2;n',\sigma')^*\\
&=\mathscr{F}_0(\lambda-\kappa^2;n,\sigma)\,v\left\{\mathcal{O}(\kappa)-\frac{1}{\kappa^2}\Big[\mathcal{O}(\kappa^2)+C_{20}(\kappa)\Big]S_2I_3(\kappa)^{-1}S_2\right.\\
&\qquad\left.\times\Big[\mathcal{O}(\kappa^2)+C_{20}(\kappa)\Big]\right\}v\,\mathscr{F}_0(\lambda-\kappa^2;n',\sigma')^*.
\end{aligned}$$

Therefore, since $\mathscr{F}_0(\lambda-\kappa^2;n,\sigma)\in\mathcal{O}(|\kappa|^{-1/2})$ and $\mathscr{F}_0(\lambda-\kappa^2;n',\sigma')^*\in\mathcal{O}(|\kappa|^{-1/2})$ by (3.19), and since $S_2C_{20}(\kappa)\,v\,\mathscr{F}_0(\lambda-\kappa^2;n',\sigma')^*\in\mathcal{O}(|\kappa|^{3/2})$ by the above arguments, one infers that the limit $\lim_{\kappa\to 0}\mathscr{F}_0(\lambda-\kappa^2;n,\sigma)\,v\,\mathrm{M}(\lambda,\kappa)\,v\,\mathscr{F}_0(\lambda-\kappa^2;n',\sigma')^*$ exists, and thus that the limit $\lim_{\kappa\to 0} S(\lambda-\kappa^2;n,\sigma,n',\sigma')$ exists by (3.20). ☐

We finally consider the continuity of the scattering matrix at embedded eigenvalues not located at thresholds.

Proposition 3.9. *Suppose that $V\in\mathrm{L}^\infty(\Omega;\mathbb{R})$ has bounded support, take $\lambda\in\sigma_{\mathrm{p}}(H)\setminus\tau$, $\kappa\in\partial\vartheta(\varepsilon)$ with $\varepsilon>0$ small enough, $n,n'\geq 1$, and $\sigma,\sigma'\in\{+,-\}$. Then, if $\lambda_n<\lambda$ and $\lambda_{n'}<\lambda$, the limit $\lim_{\kappa\to 0} S(\lambda-\kappa^2;n,\sigma,n',\sigma')$ exists.*

Proof. We know from (3.16) that

$$\mathrm{M}(\lambda,\kappa)=\big(J_0(\kappa)+S\big)^{-1}+\frac{1}{\kappa^2}\big(J_0(\kappa)+S\big)^{-1}SJ_1(\kappa)^{-1}S\big(J_0(\kappa)+S\big)^{-1},$$

with S the Riesz projection associated with the value 0 of the operator $T_0=u+\sum_n v\big(\mathcal{P}_n\otimes R^0(\lambda-\lambda_n)\big)\,v$. Now, a commutation of S with $\big(J_0(\kappa)+S\big)^{-1}$ gives

$$\mathrm{M}(\lambda,\kappa)=\big(J_0(\kappa)+S\big)^{-1}+\frac{1}{\kappa^2}\big\{S\big(J_0(\kappa)+S\big)^{-1}+\mathcal{O}(\kappa)\big\}SJ_1(\kappa)^{-1}S\big\{\big(J_0(\kappa)+S\big)^{-1}S+\mathcal{O}(\kappa)\big\},$$

and a computation as in the proof of Lemma 3.7(a) (but which takes directly Lemma 2.5 into account) shows that $\mathscr{F}_0(\lambda-\kappa^2;n,\sigma)\,vS\in\mathcal{O}(\kappa^2)$ and $Sv\,\mathscr{F}_0(\lambda-\kappa^2;n',\sigma')^*\in\mathcal{O}(\kappa^2)$. These estimates, together with the expansion (3.18) for $\mathscr{F}_0(\lambda-\kappa^2;n,\sigma)$ and $\mathscr{F}_0(\lambda-\kappa^2;n',\sigma')^*$ and the equation (3.20), imply the claim. ☐

References

[1] E. B. Davies. *Linear operators and their spectra.* Cambridge Studies in Advanced Mathematics 106, Cambridge University Press, Cambridge, 2007.

[2] M.B. Erdoğan and W. Schlag. *Dispersive estimates for Schrödinger operators in the presence of a resonance and/or an eigenvalue at zero energy in dimension three.* I. Dyn. Partial Differ. Equ. 1(4), 359–379, 2004.

[3] A. Galtbayar, A. Jensen and K. Yajima. *Local time-decay of solutions to Schrödinger equations with time-periodic potentials.* J. Statist. Phys. 116(1/4), 231–282, 2004.

[4] H. Isozaki and S. Richard. *On the wave operators for the Friedrichs-Faddeev model.* Ann. Henri Poincaré 13, 1469–1482, 2012.

[5] K. Ito and A. Jensen. *A complete classification of threshold properties for one-dimensional discrete Schrödinger operators*. Preprint arXiv: 1312.1396.

[6] A. Jensen and T. Kato. *Spectral properties of Schrödinger operators and time-decay of the wave functions.* Duke Math. J. 46(3), 583–611, 1979.

[7] A. Jensen and G. Nenciu. *A unified approach to resolvent expansions at thresholds.* Rev. Math. Phys. 13(6), 717–754, 2001.

[8] A. Jensen and G. Nenciu. *Erratum: "A unified approach to resolvent expansions at thresholds"*. Rev. Math. Phys. 16(5), 675–677, 2004.

[9] T. Kato. *Perturbation theory for linear operators*. Reprint of the 1980 edition. Classics in Mathematics. Springer-Verlag, Berlin, 1995.

[10] B. Simon. *Trace ideals and their applications*. Volume 120 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, second edition, 2005.

[11] R. Tiedra de Aldecoa. *Time delay and short-range scattering in quantum waveguides.* Ann. Henri Poincaré 7(1), 105–124, 2006.

[12] D.R. Yafaev. *Mathematical scattering theory, general theory.* Translations of Mathematical Monographs 105, American Mathematical Society, Providence, RI, 1992.